The Peter–Weyl Theorem and its Praxis

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1. Introduction

Our story begins, as many do, with Fourier analysis. Fourier analysis tells us that any complex-valued, periodic fuction that is square-integrable can be expressed as a (possibly infinite) sum of trigonometric polynomials. In more advanced language, this says that the characters (continuous homomorphisms) $\chi : S^1 \to \mathbb{C}^*$ form an orthonormal basis of $L^2(S^1)$, where it is not difficult to show that the characters of S^1 are precisely those characters $\chi : S^1 \to \mathbb{C}^*$ taking $z \mapsto z^n$ for $n \in \mathbb{Z}$. Hence, Fourier analysis elicits a deep connection between the complex-valued, square-integrable functions on the compact abelian group S^1 and its representation theory.

There are two natural ways to generalize this connection, though they are unified via the theory of harmonic analysis on locally compact groups. The first generalization is to take the results of classical Fourier analysis and see how far they can be stretched. This gives rise to the beautiful theory of Fourier analysis on locally compact abelian groups, which is nicely laid out in [2, 5]. In this theory, the groups are required to be abelian, but we are allowed to weaken the topological constraints on the group.

The second generalization, to general compact groups, allows us to soften the requirement that the group be abelian at the expense of topological generality. This is the subject of this paper; our goal is to describe the connection between the representations of compact groups and their complex-valued, square-integrable functions. The crux of this theory is the so-called *Peter–Weyl theorem*, which makes the aforementioned connection precise.

The first part of the paper is devoted to developing the machinery needed to prove the Peter–Weyl theorem (modulo some hand-waving, which we do for the sake of brevity). In particular, the proof of the theorem requires understanding the representation theory of compact groups, which we develop in Section 3, as well as some results from functional analysis, which we state in Sections 2 and 4. We prove Peter–Weyl in Section 5. The second part illustrates the manifold consequences of this theorem to various aspects of Lie group theory. In Section 6, we state (and, in some cases, prove) some deep but immediate corollaries of the Peter–Weyl theorem. Afterwards, in Section 7, we illustrate an application of this theory to SU(2), the Lie group of 2×2 unitary matrices of determinant 1.

What follows is mostly adapted from Part I of [1] and Chapters 1, 2, 3, and 5 of [2]. We also rely on [4] in our discussion of Haar measure.

2. Haar Measure

In the following, let G denote a compact group unless specified otherwise. The first order of business is to endow G with a measure so that we may integrate over the group. Since we are studying G as a topological group, not solely a topological space, we would like to give G a measure that interacts particularly nicely with the group structure on G. Such a measure is called *Haar measure*.

For any locally compact Hausdorff topological group $(X, \cdot)$, a *left Haar measure* on X is a regular Borel measure μ_L that is invariant under multiplication on the left. In other words, for any Borel set $S \subset X$, we have that $\mu_L(xS) = \mu_L(S)$ for any $x \in X$. For any compact subset K of X, one can show that $\mu_L(K) < \infty$; for any open subset U of X, one can show that $\mu_L(U) > 0$. Our stipulation that μ_L be left-translation invariant is entirely arbitrary, as one may define a right Haar measure μ_R on X similarly. It turns out that if a Haar measure—left or right—exists, then it is unique up to multiplication by some positive real constant. It is important to note that left and right Haar measures do not necessarily coincide. If they do, then X is called *unimodular*. Compact groups are particularly nice because every compact group is unimodular, and the entire group has finite measure (Proposition 1.1 of [1]). Thus, if X is compact, we need not make a distinction between left and right Haar measure, and, normalizing, we may assume the Haar measure of the entire group X is 1. Proving the existence and uniqueness of Haar measure in general requires heavy duty functional analysis that is beyond the scope of this paper. For a proof of these results, we refer the reader to [2], while [3] offers an introduction to the requisite analytical ideas.

We will, however, discuss the existence of Haar measure on compact Lie groups, which follows from the theorem below:

Theorem 2.1. *Let G be a Lie group of dimension n with a left-invariant orientation. Then G carries a positively oriented left-invariant n-form ω_G.*

Proof. Let $e_1, \dots, e_n$ be a basis for the Lie algebra of G, i.e., a basis for the tangent space of G at the identity. Recall that $e_1, \dots, e_n$ is left-invariant: if L_g denotes left-translation by $g \in G$, then $L_{g_*} e_i = e_i$. Without loss of generality, we may assume that the $e_1, \dots, e_n$ is positively oriented (otherwise, replace e_1 by $-e_1$). Let $\epsilon_1, \dots, \epsilon_n$ denote the corresponding cotangent vectors. We see that

$$(L_g^* \epsilon_i)(e_j) = \epsilon_i(L_{g_*} e_j) = \epsilon_i(e_j) = \delta_{ij},$$

so the ϵ_i's are left-invariant. We set $\omega_G = \epsilon_1 \wedge \cdots \wedge \epsilon_n$. Note that this is a left-invariant n-form, since

$$L_g^*(\omega_G) = L_g^* \epsilon_1 \wedge \cdots \wedge L_g^* \epsilon_n = \epsilon_1 \wedge \cdots \wedge \epsilon_n = \omega_G.$$

We have $\omega_G(e_1, \dots, e_n) = 1 > 0$, so ω_G is an orientation form for the orientation we fixed earlier. Thus, ω_G is a positively oriented left-invariant n-form ω_G. □

In short, for any n-dimensional Lie group G, we have constructed a positively oriented, left-invariant n-form ω_G. This is called the *Haar volume form* on G, and it induces a measure μ on Borel subsets of G by

$$\mu(B) = \int_B \omega_G.$$

It is easily verified that μ is left-translation invariant:

$$\mu(gB) = \int_{gB} \omega_G = \int_B L_g^*(\omega_G) = \int_B \omega_G.$$

Thus, by the uniqueness of Haar measure, it follows that the Haar measure on a Lie group is the measure μ induced by this positively oriented, left-invariant n-form ω_G.

3. Schur Things

In this section, we develop the representation theory of compact groups requisite to understanding the statement and proof of the Peter–Weyl Theorem. Because compact groups can be thought of as the topological-group analogs of finite groups (indeed, finite groups are compact with respect to the discrete topology), much of the representation theory is nearly the same as in the finite case. We will rely on this analogy heavily to abridge some of the proofs, but we will be careful to note where the theories diverge.

As usual, let G be a compact group with Haar measure μ, normalized so that $\mu(G) = 1$. Recall that a *representation* of G is a continuous homomorphism $\rho : G \to \mathrm{GL}(V)$, where V is some finite-dimensional, complex vector space. We could, more generally, take V to be some Hilbert space and ask for a homomorphism to the unitary group of V, but our focus from here on out will be on finite-dimensional, complex representations unless we specify otherwise. Let the character of ρ be denoted χ_ρ.

Recall that two key properties of representations of finite groups are *complete reducibility*—that every representation of a finite group is a direct sum of irreducible representations—and *Schur's lemma*—that every equivariant map between two irreducible representations is either an isomorphism or 0. These results hold for representations of compact groups as well.

Let (ρ, V) be a representation of G. If V carries an inner product $\langle\,,\rangle$ (i.e., a positive-definite Hermitian form), then we say that $\langle\,,\rangle$ is G-equivariant if it is invariant under the G-action, i.e., if $\langle \rho(g)v, \rho(g)w\rangle = \langle v, w\rangle$. As in the finite dimensional case, given any Hermitian product $(\,,)$ on V, we can construct an inner product on V given by

$$\langle v, w\rangle = \int_G (\rho(g)v, \rho(g)w)\mathrm{d}\mu(g).$$

It is not difficult to check that $\langle\,,\rangle$ is G-equivariant. From this, we deduce complete reducibility; we omit the proof since it is exactly the same as in the finite case.

Theorem 3.1. *Every finite-dimensional representation of a compact group G can be written as the direct sum of irreducible representations.*

For two representations (ρ, V) and (π, W) of G, a G-equivariant map (or G-module homomorphism) $V \longrightarrow W$ is often called an *intertwining operator*. As usual, let $\mathrm{Hom}_{\mathbb{C}}(V, W)$ denote the space of linear maps $V \longrightarrow W$; let $\mathrm{Hom}_G(V, W)$ be the subspace of intertwining operators. The proof of the following is exactly the same as in the setting of finite groups.

Theorem 3.2 (Schur's lemma). *Let (ρ, V) and (π, W) be irreducible representations of G; let $T \in$ $Hom_G(V, W)$. Then T is either an isomorphism or 0. Furthermore, if $(\pi, W) = (\rho, V)$, then there exists a scalar $\lambda \in \mathbb{C}$ such that $T = \lambda \cdot \mathrm{id}$.*

Now, we introduce the central objects of study in this representation theory, which are generalizations of characters:

Definition 3.3. A *matrix coefficient* on G is a function $\phi : G \longrightarrow \mathbb{C}$ given by $\phi(G) = \ell(\rho(g)v)$, where (ρ, V) is some representation and $\ell \in V^*$ is a linear functional $\ell : V \longrightarrow \mathbb{C}$.

By choosing a basis $v_1, \ldots, v_n$ of V, we identify V with $\mathbb{C}^n$ and can write $\rho(g)$ using a matrix. If $v = \sum_i c_i v_i$ and $\rho(g)(v_j) = \sum_i \rho_{ij}(g)v_i$, then

$$\rho(g)(v) = \begin{bmatrix} \rho_{11}(g) & \cdots & \rho_{1n}(g) \\ \vdots & \ddots & \vdots \\ \rho_{n1}(g) & \cdots & \rho_{nn}(g) \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix},$$

and each of the functions $\rho_{ij} : G \longrightarrow \mathbb{C}$ is a matrix coefficient, since $\rho_{ij}(g) = v_i^*(\rho(g)v_j)$. Hence, the moniker "matrix coefficient" is justified. By considering sums and products of matrix coefficients (and direct sums and tensor products of representations), it is not difficult to verify that the matrix coefficients on G form a ring (Proposition 2.3 of [1]). Matrix coefficients are continuous because representations are. For (ρ, V) a representation, call any linear combination of matrix coefficients of the form $\ell(\pi(g)v)$ with $v \in V$ and $\ell \in V^*$ a matrix coefficient of the representation ρ. These linear combinations form a vector space $\mathcal{E}_\rho$ of dimension at most $\dim(V \times V^*) = \dim(V)^2$.

For some representation (ρ, V) with inner product $\langle\,,\rangle$ on V, because V is finite-dimensional, every element of V^* is of the form $\langle -, v\rangle$ for some $v \in V$.[1] This allows us to index every matrix

[1]The generalization of this statement to arbitrary Hilbert spaces is known as the *Riesz representation theorem.* We refer the reader to [3] for more details.

coefficient by a pair of elements in V: every matrix coefficient is of the form $\phi_{u,v}(g) = \langle \rho(g)u, v\rangle$ for $u, v \in V$.

Now, recall that our ultimate goal is to study $L^2(G)$, i.e., the measurable functions $f : G \to \mathbb{C}$ such that

$$\|f\|_2 = \left(\int_G |f(g)|^2 \mathrm{d}\mu(g)\right)^{1/2} < \infty.$$

Recall that $L^2(G)$ forms a Hilbert space with inner product

$$\langle f_1, f_2\rangle_2 = \int_G f_1(g)\overline{f_2(g)}\mathrm{d}\mu(g)$$

and corresponding norm $\|f\|_2 = \sqrt{\langle f, f\rangle_2}$. Note that because the matrix coefficients of G are continuous, they are therefore elements of $L^2(G)$.

In the setting of finite groups, recall that the *characters* of irreducible representation representations of a finite group form an orthonormal basis for the set of complex-valued class functions on the group. Ultimately, our goal is to show the analogous statement for $L^2(G)$, where the set of matrix coefficients takes the place of the set of characters. In fact, every character of a representation of a compact group is itself a matrix coefficient of the representation. To see this, let (ρ, V) be a representation of G and χ its character. If $v_1, \ldots, v_n$ is a basis of V with corresponding dual basis $v_1^*, \ldots, v_n^*$, then $\chi(g) = \sum_{i=1}^n v_i^*(\pi(g)v_i)$.

Many of the results on characters of representations of finite groups—in particular, the orthonormality of characters of distinct irreducible representations—are easily generalized to characters of representations of compact groups. The proofs are basically exactly the same, but summing over the group is replaced by integrating over the group. We refer the interested reader to the second section of Part I of [1], where these results are stated and proved in full.

It turns out that, in $L^2(G)$, the matrix coefficients of nonisomorphic representations of G are orthogonal. In fact, we can exhibit an orthonormal basis for $\mathcal{E}_\rho$.

Theorem 3.4 (Schur Orthogonality). *Let (V, ρ) and (W, π) be irreducible representations of G. Consider $\mathcal{E}_\rho$ and $\mathcal{E}_\pi$ as subspaces of $L^2(G)$. If $\pi \neq \rho$, then $\mathcal{E}_\rho$ is orthogonal to $\mathcal{E}_\pi$. Moreover, if $v_1, \ldots, v_n$ is an orthonormal basis of V and $\rho_{ij} = \phi_{v_j, v_i}$, then $\sqrt{\dim(V)}\rho_{ij}$ for $1 \le i, j \le \dim(V)$ is an orthonormal basis for $\mathcal{E}_\rho$.*

Proof. Throughout we use $\langle , \rangle$ to denote G-equivariant forms on V and W. Suppose f_ρ and f_π are two matrix coefficients on ρ and π, respectively. We will construct an intertwining operator $T : V \longrightarrow W$, and apply Schur's lemma (Theorem 3.2) to prove the theorem. Recall from our discussion above that $f_\rho(g) = \langle \rho(g)v', v\rangle$ for some $v, v' \in V$ and that $f_\pi(g) = \langle \pi(g)w', w\rangle$ for some $w, w' \in W$. For these fixed $v \in V$ and $w \in W$, define a map $T : V \longrightarrow W$ by

$$T(x) = \int_G \langle \rho(g)x, v\rangle \pi(g^{-1})w\mathrm{d}\mu(g).$$

We see that T is in fact an intertwining operator: for any $h \in G$, we have

$$T(\rho(h)x) = \int_G \langle \rho(gh)x, v\rangle \pi(g^{-1})w\mathrm{d}\mu(g) = \int_G \langle \rho(g)x, v\rangle \pi(hg^{-1})w\mathrm{d}\mu(g) = \pi(h)T(x),$$

where the second equality follows from the translation-invariance of Haar measure after making the change of variables $g \mapsto gh^{-1}$. We have that

$$\langle f_\rho, f_\pi\rangle_2 = \int_G \langle \rho(g)v', v\rangle \overline{\langle \pi(g)w', w\rangle}\mathrm{d}\mu(g).$$

Hence,

$$\langle T(v'), w'\rangle = \int_G \langle \rho(g)v', v\rangle\langle \pi(g^{-1})w, w'\rangle \mathrm{d}\mu(g) = \int_G \langle \rho(g)v', v\rangle\overline{\langle \pi(g)w', w\rangle}\mathrm{d}\mu(g) = \langle f_\rho, f_\pi\rangle_2,$$

where the second equality follows from the fact that $\langle\,,\rangle$ is an invariant inner product on W. Applying Schur's lemma, we see that if $\pi \not\simeq \rho$, then $T = 0$ and $\mathcal{E}_\rho$ is orthogonal to $\mathcal{E}_\pi$.

We actually show something slightly stronger than the second statement. With notation as in the above, suppose that $\rho \simeq \pi$. If v and w are fixed, then define T as in the above. By Schur's lemma, there exists some constant $c(v, w)$ such that $T = c(v, w) \cdot \mathrm{id}$. Thus,

$$\langle Tv', w'\rangle = \int_G \langle \rho(g)v', v\rangle\overline{\langle \rho(g)w', w\rangle}\mathrm{d}\mu(g) = c(v, w)\langle v', w'\rangle.$$

Analogously, we can show that there exists another constant $c'(v', w')$ such that the above is equal to $c'(v', w')\langle w, v\rangle$. It follows that there must be some constant d, independent of v, v', w, and w', such that

$$\int_G \langle \rho(g)v', v\rangle\overline{\langle \rho(g)w', w\rangle}\mathrm{d}\mu(g) = \frac{\langle v', w'\rangle\langle w, v\rangle}{d}. \tag{1}$$

We will show that $d = \dim(V)$; in conjunction with (1), this will imply the theorem. Recall that $v_1, \ldots, v_n$ denotes an orthonormal basis of V, and let χ denote the character of ρ. Since $\langle \rho(g)v_i, v_i\rangle$ is the (i, i)-component of the matrix representing $\rho(g)$ with respect to this basis, we have

$$\chi(g) = \sum_{i=1}^{n} \langle \rho(g)v_i, v_i\rangle.$$

By Schur orthogonality of characters, we have that

$$1 = \int_G |\chi(g)|^2 \mathrm{d}\mu(g) = \sum_{i,j} \int_G \langle \rho(g)v_i, v_i\rangle\overline{\langle \rho(g)v_j, v_j\rangle}\mathrm{d}\mu(g) = \sum_{i,j} \frac{\delta_{ij}}{d} = \frac{n}{d},$$

where the second-to-last equality follows from (1). □

4. Just Enough Analysis to be Functional

To prove Peter–Weyl, we will need some basic facts from functional analysis, none of which we will prove. The proofs of these facts can be found in Chapter 3 of [1], in [3], or in [2].

Let $\mathfrak{H}$ be a Hilbert space. We define a norm on the space of *bounded* linear operators on $\mathfrak{H}$. An operator $T : \mathfrak{H} \to \mathfrak{H}$ is called *bounded* (alternatively, continuous) if there exists some constant C such that $|Tx| \leq C|x|$ for all $x \in \mathfrak{H}$. If we take $\mathfrak{H}$ to be a Hilbert space, then a bounded operator is said to be *self-adjoint* if we have $\langle Tf, g\rangle = \langle f, Tg\rangle$ for all $f, g \in \mathfrak{H}$. We call a bounded operator *compact* if for any bounded sequence $(x_n)_n$ in $\mathfrak{H}$ the sequence $(Tx_n)_n$ has a convergent subsequence. The following is the main theorem of consequence:

Theorem 4.1 (The Spectral Theorem)**.** *If T is a self-adjoint compact operator on a Hilbert space $\mathfrak{H}$, then there is an orthonormal basis of $\mathfrak{H}$ consisting of eigenvectors for T.*[2]

[2]Recall that because our Hilbert space is potentially infinite, a basis for $\mathfrak{H}$ is a linearly independent subset B of $\mathfrak{H}$ such that the closure of the span of B is all of $\mathfrak{H}$.

We will also need Arzelà–Ascoli, which is a fundamental result in the study of continuous functions on a compact space. For a compact space X, let $C(X)$ denote the complex-valued continuous functions on X. Topologize $C(X)$ using the sup norm $\|\cdot\|_\infty$. Recall that a subset $U \subset C(X)$ is said to be *equicontinuous* if for every $x \in X$ and $\epsilon > 0$, there exists a neighborhood A of x such that $|f(x) - f(y)| < \infty$ for all $y \in A$ and all $f \in U$.

Theorem 4.2 (Arzelà–Ascoli)**.** *Let X be a compact Hausdorff space; let $U \subset C(X)$ be a uniformly bounded, equicontinuous subset. Then the closure of U in $C(X)$ is compact.*

Finally, we introduce the notion of convolution. As usual, let G be a compact group. We define an operation on $L^1(G)$ (the set of complex-valued integrable functions) called *convolution*, given by

$$(f_1 * f_2)(g) = \int_G f_1(gh^{-1})f_2(h)\mathrm{d}\mu(h) = \int_G f_1(h)f_2(h^{-1}g)\mathrm{d}\mu(h)$$

(the last equality is given by making the change of variables $h \mapsto h^{-1}g$ and making use of the unimodularity of G). One can think of convolution as taking a weighted average of one function against the other. Convolution has many deep implications, one of which is that it turns $L^1(G)$ into a (nonunital) ring. We will make use of the following two theorems:

Theorem 4.3. *Suppose G is unimodular. If $f_1, f_2 \in L^2(G)$, then $f_1 * f_2$ is a continuous function on G vanishing at infinity, and $\|f_1 * f_2\|_\infty \le \|f_1\|_2\|f_2\|_2$ (here $\|\cdot\|_\infty$ denotes the sup norm rather than the norm on $L^\infty(G)$).*

Theorem 4.4. *If G is compact, then $C(G)$ has an* approximate identity*. In particular, let $\mathcal{U}$ be a neighborhood base at 1 in G. For each open $U \in \mathcal{U}$, let ψ_U be a real-valued function such that the following hold:*

(1) ψ_U is compactly supported and $\operatorname{im}(\psi_U) \subset U$;
(2) $\psi_U \ge 0$ and $\int_G \psi_U = 1$;
(3) $\psi_U(x^{-1}) = \psi_U(x)$ for all x.

Then for $f \in L^2(G)$, we have that

$$\|f * \psi_U - f\|_2 \to 0 \quad as \quad U \to \{1\}.$$

5. The Peter–Weyl Theorem

Before stating the main theorem, we begin with some notation. Let $\widehat{G}$ be the set of equivalence classes of irreducible representations of G. Denote the equivalence class of a representation π by $[\pi]$. Recall that $\mathcal{E}_\pi$ consists of linear combinations of matrix coefficients of π. Let

$$\mathcal{E} = \operatorname{span}_{\mathbb{C}}\left(\bigcup_{[\pi]\in\widehat{G}} \mathcal{E}_\pi\right)$$

be the set of all finite linear combinations of matrix coefficients of irreducible representations. In the analogy with Fourier analysis, $\mathcal{E}$ takes the place of the trigonometric polynomials. Finally, recall that the right regular representation of G is the representation of G given by translation on the right.

To ease notation, for each $[\pi] \in \widehat{G}$, choose some representative $\pi \in [\pi]$. In the following, when we refer to some specific representative of $[\pi]$, we assume that we are referring to this specific choice.

Theorem 5.1 (The Peter–Weyl Theorem). *Let G be a compact group. Then $\mathcal{E}$ is uniformly dense in $C(G)$ and $L^2(G)$. Moreover, we can write*

$$L^2(G) = \bigoplus_{[\pi]\in\widehat{G}} \mathcal{E}_\pi.$$

For $[\pi] \in \widehat{G}$, if π_{ij} is as in Theorem 3.4 for $1 \le i, j \le d_\pi$, where d_π denotes the dimension of π, then

$$\left\{\sqrt{d_\pi}\pi_{ij} \mid 1 \le i, j \le d_\pi, [\pi] \in \widehat{G}\right\}$$

forms an orthonormal basis of $L^2(G)$.

Proof. Let $\{\psi_U\}$ be an approximate identity as in Theorem 4.4. Fix some U, and let $\psi = \psi_U$. For $f \in L^2(G)$, set $T_\psi f = \psi * f$. We claim that T_ψ is self-adjoint on $L^2(G)$. To see why this is the case, note that

$$\langle T_\psi f_1, f_2\rangle_2 = \int_G (\psi * f_1)(g)\overline{f_2(g)}\mathrm{d}\mu(g) = \int_G\int_G \psi(gh^{-1})f_1(h)\overline{f_2(g)}\mathrm{d}\mu(h)\mathrm{d}\mu(g)$$

and that

$$\langle f_1, T_\psi f_2\rangle_2 = \int_G\int_G f_1(h)\overline{\psi(hg^{-1})f_2(g)}\mathrm{d}\mu(g)\mathrm{d}\mu(h).$$

By Fubini's theorem and the fact that ψ is real and symmetric ($\psi(x^{-1}) = \psi(x)$ for $x \in G$), we have that $\langle T_\psi f_1, f_2\rangle_2 = \langle f_1, T_\psi f_2\rangle_2$, as desired. Applying Theorem 4.3, we see that T_ψ is a map $L^2(G) \to C(G)$. Moreover, the same result implies that $\|T_\psi f\|_\infty \le \|f\|_2\|\psi\|_2$. For a function a on G, let $L_x a(g) = a(x^{-1}g)$ (define R_x similarly). Theorem 4.3 further implies that

$$\|L_x(T_\psi f) - T_\psi f\|_\infty = \|(L_x\psi - \psi) * f\|_\infty \le \|f\|_2\|L_x\psi - \psi\|_2.$$

Let B be a bounded set in $L^2(G)$, and consider the set

$$\{T_\psi f \mid f \in B\}.$$

The first inequality $\|T_\psi f\|_\infty \le \|f\|_2\|\psi\|_2$ implies that $\{T_\psi f \mid f \in B\}$ is uniformly bounded; the second inequality implies that $\{T_\psi f \mid f \in B\}$ is equicontinuous. Hence, the hypotheses of Arzelà–Ascoli (Theorem 4.2) are satisfied, and we conclude that T_ψ is compact as a map $T_\psi : L^2(G) \to C(G)$. It follows *a fortiori* that T_ψ is a compact operator on $L^2(G)$.

Thus, we may apply the spectral theorem (Theorem 4.1) to T_ψ, which allows us to write

$$L^2(G) = \bigoplus_\alpha \mathscr{M}_\alpha,$$

where the sum runs over α the eigenvalues of T_ψ and $\mathscr{M}_\alpha$ denotes the corresponding eigenspace. Next, note that $R_x(\psi * f) = \psi * R_x f$, which implies that $\mathscr{M}_\alpha$ is invariant under right-translation. For the nonzero eigenvalues α, we must have $\dim(\mathscr{M}_\alpha) < \infty$. This follows because the restriction of any compact operator to $\mathscr{M}_\alpha$ must be compact, and the map $f \mapsto \alpha f$ is compact if and only if f is finite-dimensional. Thus, take $f_1, \ldots, f_n$ be to be an orthonormal basis for $\mathscr{M}_\alpha$. Define $\rho_{jk}(x) = \langle R_x f_k, f_j\rangle$, and note that $f_k(yx) = \sum_j \rho_{jk}(x)f_j(y)$. It follows that $f_k(x) = \sum_j f_j(1)\rho_{jk}(x)$. In other words, each f_k is a linear combination of the matrix coefficients of the regular representation ρ of $\mathscr{M}_\alpha$. Therefore, $\mathscr{M}_\alpha \subset \mathcal{E}_\rho \subset \mathcal{E}$ for each nonzero α.

Recall from our application of the spectral theorem that any $f \in L^2(G)$ can be written as a convergent (in $L^2(G)$) series $f = \sum_\alpha f_\alpha$ for $f_\alpha \in \mathscr{M}_\alpha$. Because $T_\psi : L^2(G) \to C(G)$ is bounded, it follows that we may write $T_\psi = \sum_{\alpha\ne 0} \alpha f_\alpha$, where the series converges uniformly. Moreover, because we have that each $\mathscr{M}_\alpha \subset \mathcal{E}$, we have that $\mathcal{E} \cap T_\psi(L^2(G))$ is uniformly dense in $T_\psi(L^2(G))$.

Finally, because $\{\psi_U\}$ was chosen to approximate identity, it follows from Theorem 4.4 that

$$\bigcup_U T_\psi(L^2(G))$$

is (uniformly) dense in $C(G)$. It follows that $\mathcal{E}$ is dense in $C(G)$, which is itself dense in $L^2(G)$. Thus, $\mathcal{E}$ is dense in $L^2(G)$, and combining this with Theorem 3.4 allows us to write $L^2(G)$ as the orthogonal direct sum of $\mathcal{E}_\pi$ for all $[\pi] \in \widehat{G}$. Proceeding as in Theorem 3.4, we get an orthonormal basis for $L^2(G)$ by choosing an orthonormal basis for each representation π and considering the corresponding orthonormal basis of matrix coefficients with respect to this basis. □

6. Consequences

The Peter–Weyl theorem has manifold applications, a few of which we discuss in the following. While the results of the last two sections have been general, applying to arbitrary compact groups, there are some immediate Lie-theoretic consequences.

One of these is that every compact Lie group can be realized as a closed matrix group. Recall that every matrix group is a Lie group, since any matrix group can be embedded in GL_n for some n. However, the converse is not always true, since there exist Lie groups that admit no faithful finite-dimension representation. For example, consider the Lie group $\mathrm{SL}_2\mathbb{R}$ and its universal cover $\widetilde{\mathrm{SL}_2}\mathbb{R}$. Let Π be a finite-dimensional real representation of $\widetilde{\mathrm{SL}_2}\mathbb{R}$. Consider the associated Lie group representation $\pi = d\Pi(1)$ of $\mathfrak{sl}_2\mathbb{R}$. Taking tensor products over $\mathbb{R}$ with $\mathbb{C}$ gives a representation of $\mathfrak{sl}_2\mathbb{C}$; since $\mathrm{SL}_2\mathbb{C}$ is simply connected, this yields a representation $\Phi_\mathbb{C}$ of $\mathrm{SL}_2\mathbb{C}$. Now, recall that $\mathrm{SL}_2\mathbb{R} \subset \mathrm{SL}_2\mathbb{C}$, so $\Phi_\mathbb{C}$ restricts to a representation of $\mathrm{SL}_2\mathbb{R}$, which we call Φ. The covering map $\rho : \widetilde{\mathrm{SL}_2}\mathbb{R} \longrightarrow \mathrm{SL}_2\mathbb{R}$ then gives us another representation of $\widetilde{\mathrm{SL}_2}\mathbb{R}$, given by composing Φ with ρ. If $\phi = d(\Phi \circ \rho)(1)$ is the corresponding representation of $\mathfrak{sl}_2\mathbb{R}$, then it is straightforward to verify that $\phi = \pi$. Hence, $\Pi = \Phi \circ \rho$; since $\rho : \widetilde{\mathrm{SL}_2}\mathbb{R} \longrightarrow \mathrm{SL}_2\mathbb{R}$ is a covering map, we conclude that Π has nontrivial kernel. It follows that $\widetilde{\mathrm{SL}_2}\mathbb{R}$ is a Lie group that cannot be realized as a matrix group, since it has no finite-dimensional representations that are faithful.

The Peter–Weyl theorem gives us a partial converse, however. Recall that a topological group has *no small subgroups* if there is a neighborhood of the identity that does not contain nontrivial subgroups. To see that a Lie group G has no small subgroups, take a bounded open neighborhood V of 0 in the Lie algebra $\mathfrak{g}$ and consider $U = \exp(V/2)$. For a nontrivial element $g \in U$, then note that some positive integer power g^n lies in $\exp(V \setminus V/2) \subset G \setminus U$. Hence, no subgroup containing g can be contained in U.

Theorem 6.1. *Let G be a compact group that has no small subgroups. Then G has a faithful finite-dimensional representation.*

Proof. Since G has no small subgroups, there exists a neighborhood U of the identity that does not contain a nontrivial subgroup. Let ϕ be a continuous function with $\phi(1) = 0$ and $\phi(g) > 1$ for $g \notin U$. By the Peter–Weyl theorem, there exists a matrix coefficient f corresponding to some $[\pi] \in \widehat{G}$ such that $\|f - \phi\|_\infty < \epsilon$ for any $\epsilon > 0$. It follows that there exists a matrix coefficient f with $f(1) = 0$ and $f(g) > 1$ for $g \notin U$. Since $f(x) = \ell(\pi(x)v)$ for some v and ℓ, we note that $f(x)$ is constant on $\ker(\pi)$. Therefore, $f|_{\ker(\pi)} = 0$, implying that $\ker(\pi) \subset U$. The kernel of π is a subgroup of G; by our choice of U, it must be trivial. Hence, π is a faithful finite-dimensional representation of G. □

Corollary 6.2. *A compact group G is a Lie group if and only if it can be realized as a closed matrix group.*

As an application of the above, we note that the Peter–Weyl theorem allows us to prove that Spin(n) is a matrix group without ever referencing Clifford algebras. Recall that Spin(n) is the universal cover of SO(n). Since SO(n) is compact Lie group, so is Spin(n). It follows from Theorem 6.1 that Spin(n) can be regarded as a closed matrix group, as desired.

Another consequence of the Peter–Weyl theorem—which we will not prove—is the following theorem, which tells us that arbitrary *unitary* representations of a compact group can decomposed into a direct sum of irreducible representations. Let H be a Hilbert space with inner product $\langle\, , \rangle$. Recall that an operator $T : H \to H$ is called *unitary* if for all $u, v \in H$ we have $\langle Tu, Tv\rangle = \langle u, v\rangle$. Let $U(H)$ be the space of unitary operators on H. A unitary representation of a compact group G is the data of a Hilbert space H and a continuous homomorphism $\pi : G \to U(H)$. Complete reducibility of unitary representations is a consequence of the Peter–Weyl theorem (Theorem 5.1).

Theorem 6.3 (Theorem 4.3 of [1])**.** *Let $\pi : G \to U(H)$ be a unitary representation of a compact group G. Then H decomposes into a direct sum of finite-dimensional irreducible representations.*

Finally, we conclude with a version of Peter–Weyl for class functions (i.e., functions constant on conjugacy classes of G) on a compact group G. While we omit the proofs of these facts, we note that the proofs mostly follow easily from Theorem 5.1 and some basic Fourier analysis. For any space $\mathcal{F}$ of functions on G, let $Z\mathcal{F}$ denote the set of class functions in $\mathcal{F}$.

Theorem 6.4. *The linear span of the characters χ_π for $[\pi] \in \widehat{G}$ is dense in $ZC(G)$ and also in $L^2(G)$. Moreover, $\{\chi_\pi \mid [\pi] \in \widehat{G}\}$ is an orthonormal basis for $ZL^2(G)$.*

7. Spherical Harmonics

Lastly, we sketch an application of Theorem 5.1 to the Lie group SU(2). For a more complete version of the following results, we refer the reader to Section 4 of Chapter 5 in [2]. Recall that U(n) is the group of unitary transformations of $\mathbb{C}^n$, i.e., operators $T : \mathbb{C}^n \to \mathbb{C}^n$ such that $T^*T = \mathrm{id}_n$, where T^* is the conjugate transpose of T. The subgroup SU(n) consists of those unitary operators of determinant 1. Thus, after some simple algebra, we can write

$$\mathrm{SU}(2) = \{U_{a,b} \mid |a|^2 + |b|^2 = 1\},$$

where

$$U_{a,b} = \begin{bmatrix} a & b \\ -\overline{b} & \overline{a} \end{bmatrix}.$$

Note that $U^*_{a,b} = U^{-1}_{a,b}$ and that the above gives a correspondence between elements of SU(2) and the elements of $S^3 \subset \mathbb{C}^2$, where $U_{1,0}$ corresponds to the north pole. In fact, we can realize the Peter-Weyl theorem analytically as the decomposition of $L^2(S^3)$ into *spherical harmonics*, which will be made precise later.

To do so, we exhibit a collection of representations of SU(2). Let $\mathcal{P} = \mathbb{C}[z, w]$ be the vector space of polynomials in two complex variables, z and w, with coefficients in $\mathbb{C}$, and let $\mathcal{P}_m$ denote the vector subspace of homogeneous polynomials of degree m. Since each polynomial can be thought of as a function in two complex variables, we may view an element of $\mathcal{P}$ as an element of $L^2(S^3)$, where integration is done with respect to σ, the surface measure on the unit sphere such that $\sigma(S^3) = 1$. On each $\mathcal{P}_m$, we have a complete inner product given by

$$\langle P, Q\rangle = \int_{S^3} P\overline{Q}\mathrm{d}\sigma.$$

Some integration (simple, explicit computations which we omit), shows that the monomials $z^j w^k$ are orthogonal in $\mathcal{P}$ with respect to this inner product. We can also compute that

$$\langle z^j w^k, z^j w^k \rangle = \frac{j!k!}{(j+k+1)!},$$

which allows us to normalize the monomials $z^j w^{m-j}$ into an orthonormal basis for $\mathcal{P}_m$.

Now, there is a natural action of SU(2) on $\mathcal{P}$ induced by the action of SU(2) on $\mathbb{C}^2$: for $P \in \mathcal{P}$,

$$(U_{a,b} \cdot P)(z,w) = P(U_{a,b}^{-1}(z,w)) = P(\overline{a}z - bw, \overline{b}z + aw).$$

Let π denote the corresponding representation. It is easily verified that the SU(2)-action preserves $\mathcal{P}_m$; let π_m denote the corresponding representation. Some more work shows the following:

Theorem 7.1. *Each π_m is irreducible for $m \geq 0$. Moreover, we have that $\widehat{\mathrm{SU}}(2) = \{\pi_m \mid m \in \mathbb{N}\}$.*

By the Peter–Weyl theorem, we have that $L^2(\mathrm{SU}(2))$ decomposes as

$$\mathrm{SU}(2) = \bigoplus_{m=0}^{\infty} \mathcal{E}_{\pi_m}. \tag{2}$$

But what are the matrix coefficients? Recall that we can write explicitly the matrix coefficients of π_m with respect to the aforementioned orthonomal basis of $\mathcal{P}_m$:

$$e_j(z,w) = \sqrt{\frac{(m+1)!}{j!(m-j)!}} z^j w^{m-j}.$$

Setting $\pi_m(U_{a,\overline{b}}) = \pi(a,b)$, recall that the matrix coefficient $\pi_m^{jk}(a,b)$ is given by $\langle \pi_m(a,b)e_k, e_j \rangle$. More computation tells us that

$$\sum_j \sqrt{\frac{k!(m-k)!}{j!(m-j)!}} \pi_m^{jk}(a,b) z^j w^{m-j} = (\overline{a}z - \overline{b}w)^k (bz + aw)^{m-k}, \tag{3}$$

from which we can show that

$$\pi_m^{jk}(a,b) = \sqrt{\frac{j!(m-j)!}{k!(m-k)!}} \int_0^1 (\overline{a}e^{2\pi i t} - \overline{b})(be^{2\pi i t + a})^{m-k} e^{-2\pi i j t} \mathrm{d}t.$$

From (3), we see that $\pi_m^{jk}(a,b)$ is polynomial in $a, b, \overline{a}, \overline{b}$ that is homogeneous of degree $m-k$ in (a,b) and homogeneous of degree k in $(\overline{a}, \overline{b})$. Writing $a = x_1 + ix_2$ and $b = x_3 + ix_4$, we can verify that π_m^{jk} is *harmonic*, i.e., that

$$\frac{\partial \pi_m^{jk}}{\partial x_1^2} + \frac{\partial \pi_m^{jk}}{\partial x_2^2} + \frac{\partial \pi_m^{jk}}{\partial x_3^2} + \frac{\partial \pi_m^{jk}}{\partial x_4^2} = 4\frac{\partial \pi_m^{jk}}{\partial a \partial \overline{a}} + 4\frac{\partial \pi_m^{jk}}{\partial b \partial \overline{b}} = 0.$$

It is a well-known analytic fact that

$$L^2(S^n) = \bigoplus_{m=0}^{\infty} \mathcal{H}_k^n,$$

where $\mathcal{H}_k^n$ denotes the space of *spherical harmonics* of degree k. For $n = 1$, this decomposition is realized by Fourier analysis. Since the matrix coefficients of SU(2) are harmonic, the identification $\mathrm{SU}(2) \simeq S^3$ along with (2) tells us that the Peter-Weyl decomposition of $L^2(\mathrm{SU}(2))$ is exactly the decomposition of $L^2(S^3)$ into spherical harmonics.

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